CAYLEY'S DECOMPOSITION AND POLYA'S W-PROPERTY OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS(1)

BY

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ABSTRACT

A proof is given for the equivalence of Pólya's W- property of a linear differential equation $L_n(D)y = 0$ to the possibility of decomposing $L_n(\mathrm{D}) \equiv \Pi_n^1 [\mathrm{D} + \lambda_i(x)]$ in a given interval. In this case a set of n independent solutions form a Chebyshev system in the interval. An application determines intervals of non-oscillation for solutions of linear equations of the second order.

1. **Introduction.** In 1886 Cayley [1] considered the problem of decomposing a linear differential operator

$$(1) \qquad L_n(D) \equiv D^n + \gamma_1(x)D^{n-1} + \cdots + \gamma_n(x); \qquad D \equiv d/dx$$

into a product of linear factors

$$(2) \qquad L_n(D) \equiv [D + \lambda_n(x)][D + \lambda_{n-1}(x)] \cdots [D + \lambda_1(x)].$$

By carrying out the multiplications and differentiations in (2) and comparing coefficients with (1) he arrived at a set of differential equations for the functions $\lambda_k(x)$, $k = 1, \cdots, n$ in terms of the coefficients $\gamma_k(x)$, $k = 1, \cdots, n$. He also indicated methods for finding local solutions.

The example $D^2 + 1 \equiv [D - \tan(x + \alpha)][D + \tan(x + \alpha)]$ illustrates however the fact that a decomposition (2) does not necessarily exist in an arbitrary given interval, (e.g. one whose length exceeds π) and is not unique when it does exist there.

In 1922 Pólya [4] defined a W-property of an ordinary linear differen- equation $L_n(D)y = 0$ in a closed interval $[a, b]$ as the property of possessing n independent solutions $h_1(x), \cdots, h_n(x)$ such that the Wronskians

$$W_k = W(h_1, \cdots, h_k) \equiv \det \left| h_i^{(j)}(x) \right|_{i=1,\ldots,k;\ j=0,\ldots,k-1}$$

are all positive in $[a, b]$ for $k = 1, \cdots, n$.

Pólya has shown that the W-property of $L_n(D) = 0$ is equivalent to the possibility of writing $L_n(D)$ as a product in the following way:

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$$L_n(D) \equiv (W_n/W_{n-1})D(W_{n-1}^2/W_{n-2}W_n)D \cdots D(1/W_1).$$

It is known (compare Ince [3, p. 121]) that this form is equivalent to the decomposition (2).

In §2 we show directly that the validity of a decomposition (2) in an interval $[a,b]$ with $\lambda_k(x) \in C^{(n-k)}[a,b]$ is equivalent to Pólya's W-property of $L_n(D)y = 0$ in that interval.

In §3 we show that if $L_n(D)y = 0$ has a decomposition (2) in an interval then every set of n independent solutions form a Chebyshev (or unisolvent) system in that interval. This means that every non trivial solution has at most $n-1$ distinct zeros in that interval. Applications to Approximation Theory will appear elsewhere.

In §4 we apply the above results to the finding of intervals of non oscillation for the solutions of linear equations of the second order.

2. The equivalence between Cayley's decomposition and Pólya's W-property. Before we state our main result we prove a

LEMMA. *Suppose $L_k(D) \equiv \prod_{i=k}^{1}[D+\lambda_i]$. Then*

$$(3) \quad (D+\lambda_k+\cdots+\lambda_1)W(h_1,\cdots,h_k) = \begin{vmatrix} h_1, & \cdots, & h_k \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ h_1^{(k-2)}, & \cdots, & h_k^{(k-2)} \\ L_k(D)h_1, & \cdots, & L_k(D)h_k \end{vmatrix}$$

assuming that the functions λ_i, h_i are sufficiently differentiable to make the above expressions meaningful.

Proof. In the last row of the determinant $L_k(D)h_i$ can be replaced by $[D^k + (\lambda_k + \cdots + \lambda_1)D^{k-1}]h_i$ since the omitted terms form a linear combination of the other rows. At the same time it is easily seen that the left hand side is obtained by operating $(D+\lambda_k+\cdots+\lambda_1)$ on the last row only of the Wronskian $W(h_1,\cdots,h_k)$.

THEOREM 1. *Let $L_n(D) \equiv D^n + \gamma_1(x)D^{n-1} + \cdots + \gamma_n(x)$ be a differential operator with coefficients $\gamma_k(x)$, $k=1,\cdots,n$, continuous in an interval $[a,b]$. Then a necessary and sufficient condition that $L_n(D)$ admit in $[a,b]$ a Cayley decomposition (2) with $\lambda_k \in C^{(n-k)}[a,b]$, $k=1,\cdots,n$, is that the equation $L_n(D)y=0$ should have in $[a,b]$ Pólya's W-property i.e. there should exist n independent solutions $h_1,\cdots,h_n$ such that*

$$(4) \qquad W[h_1(x),\cdots,h_k(x)] > 0,\ x \in [a,b], \qquad k=1,\cdots,n.$$

Proof. $h_1(x) = W(h_1) = \exp[-\int_a^x \lambda_1]$ is a solution of $[D + \lambda_1(x)]y = 0$ and is also simultaneously a solution of $\prod_{i=k}^1 [D + \lambda_i(x)]y = 0$ for $k = 1, \cdots, n$. Let h_2 be a solution of

$$[D + \lambda_2(x)][D + \lambda_1(x)]y \equiv [D^2 + (\lambda_2 + \lambda_1)D + \lambda_1' + \lambda_1\lambda_2]y = 0$$

linearly independent of h_1. By Abel's identity (see [3, pp. 119]) $W(h_1, h_2) = c.\exp[-\int_a^x(\lambda_2 + \lambda_1)]$, $c \neq 0$. By suitably normalizing h_2 we may assume $c = 1$. We continue in this manner until we add to the solutions $h_1, \cdots, h_{n-1}$ of $\prod_{i=n-1}^1 [D + \lambda_i]y = 0$ a solution h_n of $\prod_{i=n}^1 [D + \lambda_i]y = 0$ independent of the previous ones and normalized so that $W(h_1, \cdots, h_n) = \exp[-\int_a^x(\lambda_n + \cdots + \lambda_1)]$. Assuming now that $L_n(D)$ has the decomposition (2), $h_1, \cdots, h_n$ are solutions of $L_n(D)y = 0$ verifying property-W. This proves the necessity of the condition in the theorem.

We shall now prove that the condition is sufficient by constructing a system of λ_k's from a set of solutions $h_1, \cdots, h_n$ of the equation

$$[D^n + \gamma_1(x)D^{n-1} + \cdots + \gamma_n(x)]y = 0 \tag{5}$$

satisfying (4). Indeed, define recursively

$$\lambda_1 = \frac{-D[W(h_1)]}{W(h_1)}; \ \lambda_k = \frac{-D[W(h_1, \cdots, h_k)]}{W(h_1, \cdots, h_k)} - \sum_1^{k-1} \lambda_i, \qquad k = 2, \cdots, n. \tag{6}$$

Denote $L_k(D) \equiv \prod_{i=k}^1 [D + \lambda_i]$, $k = 1, \cdots, n$. We shall show that $h_1, \cdots, h_i$ are solutions of $L_i(D)y = 0$ for $i = 1, \cdots, n$. The proof is by induction. The statement is true for $i = 1$ by the first part of (6). Suppose it is true for $i = k - 1$ and let us prove that it is also valid for $i = k$. By the assumption $L_k(D)h_j = L_{k-1}(D)h_j = 0$ for $j = 1, \cdots, k - 1$ and thus the right hand side of (3) in the lemma is equal to $L_k(D)h_k \cdot W(h_1, \cdots, h_{k-1})$. The left hand side of (3) vanishes by (6) and since the Wronskian is positive, $L_k(D)h_k = 0$. Hence $h_1, \cdots, h_n$ are independent solutions of the equation $L_n(D)y \equiv [\prod_{i=n}^1 (D + \lambda_i)]y = 0$ which proves its identity with the equations (5).

3. The number of zeros of a solution.

Theorems 2 and 3 are generalizations of Rolle's theorem.

THEOREM 2. *Let $f(x)$ be differentiable in the compact interval $[a,b]$ and suppose $f(a) = f(b) = 0$. Then for every continuous function $\lambda(x)$ in $[a,b]$ there exists a number c, $a < c < b$, such that*

$$[D + \lambda(c)]f(c) = f'(c) + \lambda(c)f(c) = 0. \tag{7}$$

Proof. Apply Rolle's theorem to the function $G(x) = f(x)\exp[\int \lambda(x)dx]$. Clearly $G(a) = G(b) = 0$ and $G'(x) = [f'(x) + \lambda(x)f(x)]\exp[\int \lambda(x)dx]$. (7) follows from $G'(c) = 0$.

In view of Theorem 1, the next two theorems are essentially the same as Theorems I and II of Pólya [4]. The short proofs given here are made possible by using the decomposition (2) as point of departure.

THEOREM 3. *Let $f(x)$ be n times differentiable in the interval $[a,b]$ and have there $n+1$, distinct zeros and let $\lambda_k(x)$ belong to $C^{(n-k)}[a,b]$, $k = 1,\cdots,n$. Then there exists a number c, $a < c < b$, such that*

$$f_n(c) = [D + \lambda_n(x)][D + \lambda_{n-1}(x)]\cdots[D + \lambda_1(x)]f(x)\big|_{x=c} = 0. \tag{8}$$

Proof. Let us define $f_k(x)$ for $k = 0, 1, \cdots, n$ as follows: $f_0(x) \equiv f(x); f_k(x) \equiv [D + \lambda_k(x)]f_{k-1}(x)$ for $k = 1,\cdots,n$. An inductive argument, which makes use of Theorem 2, shows that $f_k(x)$ has at least $n-k+1$ zeros, each lying between each pair of adjacent zeros among the $n-k+2$ zeros of $f_{k-1}(x)$ $(k = 1,\cdots,n)$. For $k = n$ we obtain (8).

THEOREM 4. *Let $y = f(x) \not\equiv 0$ be a function defined on the compact interval interval $[a,b]$ and satisfy there the linear differential equation*

$$L_n(D)y \equiv [D + \lambda_n(x)][D + \lambda_{n-1}(x)]\cdots[D + \lambda_1(x)]y = 0 \tag{9}$$

where n is a positive integer and $\lambda_k(x) \in C^{(n-k)}[a,b]$, $k = 1,\cdots,n$. Then $f(x)$ has at most $n-1$ distinct zeros in $[a,b]$.

The conclusion remains true if the interval $[a,b]$ is replaced by a finite or infinite open or semi-open interval.

Proof. The proof is by induction. For $n = 1$, equation (9) is reduced to $[D + \lambda_1(x)]y = 0$ whose general non-trivial solution is given by $y = c\exp[-\int \lambda(x)dx]$, $c \neq 0$, and has no zeros in $[a,b]$ as claimed. Now suppose our theorem to be true for $k = n-1$. We shall prove that it is also valid for $k = n$. Indeed, if $y = f(x) \not\equiv 0$ is a solution of equation (9), then the function

$$f_{n-1}(x) \equiv [D + \lambda_{n-1}(x)]\cdots[D + \lambda_1(x)]f(x)$$

is a solution of the equation

$$[D + \lambda_n(x)]f_{n-1}(x) = 0. \tag{10}$$

Two cases are possible. If $f_{n-1}(x)$ is a non-trivial solution of (10), then by the case $n = 1$ discussed above, it has no zero in $[a,b]$ and therefore, by Theorem 3, $f(x)$ can have at most $n-1$ zeros there. If on the other hand $f_{n-1}(x) \equiv 0$, then $f(x)$ is a solution of an equation of order $n-1$

$$[D + \lambda_{n-1}(x)] \cdots [D + \lambda_1(x)]f(x) = 0$$

and by the assumption of induction $f(x)$ has at most $n-2$ zeros in $[a,b]$.

COROLLARY. *If $h_1(x), \cdots, h_n(x)$ are linearly independent solutions of equation (9), then they form a Chebyshev system in $[a,b]$, i.e. every non linear combination $\sum_1^n a_k h_k(x)$ with $\sum_1^v |a_k| > 0$ has at most $n-1$ distinct zeros in $[a,b]$.*

4. Intervals of non oscillation of solutions. It follows from Theorem 4 that if a decomposition

$$L_2(D) \equiv D^2 + A(x)D + B(x) \equiv [D + \lambda_2(x)][D + \lambda_1(x)] \tag{11}$$

is valid in an interval $[a,b]$ then every non trivial solution of $L_2(D)y = 0$ is non-oscillatory in $[a,b]$ i.e. has at most one zero in that interval. We assume here A, B, $\lambda_2 \in C[a,b]$ and $\lambda_1 \in C^{(1)}[a,b]$. From (11) we obtain

$$\lambda_1(x) + \lambda_2(x) = A(x) \quad \text{and} \quad \lambda_1(x)\lambda_2(x) + \lambda_1'(x) = B(x). \tag{12}$$

The simultaneous solvability of (12) is equivalent to the existence in $[a,b]$ of a solution of the Ricatti equation (compare Ince [3, p. 24])

$$\lambda_1'(x) = \lambda_1(x)^2 - A(x)\lambda_1(x) + B(x).$$

We can now apply the standard Cauchy-Lipschitz existence theorem (see [2, p. 3]) to find intervals of non-oscillation:

THEOREM 5. *Let $A(x)$ and $B(x)$ be continuous in the interval $[a,b]$ and let*

$$M = M(y_0, h) = \max\left[|y^2 - A(x)y + B(x)|,\ x \in [a,b], |y - y_0| < h \right].$$

Then every non-trivial solution of

$$[D^2 + A(x)D + B(x)]y = 0$$

has at most one zero in the interval $[a,c]$, where $c - a = \min[b-a, h/M]$.

References

1. Arthur Cayley, *On linear differential equations (the theory of decomposition)*, Quarterly J. of Pure and Appl. Math., **21** (1886), 331–335.

2. Lamberto Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Springer-Verlag, Berlin, 1959.

3. E. L. Ince, *Ordinary differential equations*, Longmans, Green and Co., London, 1927.

4. Georg Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

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